Lecture Notes in Audio Analysis: Introduction to Audio Equalizers

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Abstract

This document contains a brief introduction to audio equalizer design, more specifically how to design parametric equalizer filters from simple second-order peak and notch filters and how to obtain shelving filters as special cases of the parametric equalizer filters. It requires some basic understanding of signal processing operations and concepts like filtering, z-transforms, Fourier transforms, etc. For a more in-depth treatment of the matter, we refer the interested reader to [1].

Introduction

Equalizers are ubiquitous in audio equipment, be it a simple tone control on a stereo, a hifi multi-band graphic equalizer, or a parametric equalizer band in a guitar pedal. Although based on the same principles, equalizers serve different purposes depending on their context. In reproduction systems, equalizers serve to cancel any undesired filtering effect due to the involved equipment (loud speakers, amplifiers, etc.) or room acoustics. Musicians use equalizers as a means for shaping their sound, i.e., as part of a musical expression. Equalizers are built from linear time-invariant (as long as you do not touch the controls) filters. There are two kinds of equalizers: graphic and parametric. In graphic equalizers, a desired frequency response (i.e., attenuation and amplification of certain frequencies) is achieved by changing the gain at a set of fixed center frequencies. In parametric equalizers, the desired frequency response is obtained using a number of bands, each of which can be controlled by a center frequency, a bandwidth and a gain (or other similar parameters). In the following, we will cover the design of parametric equalizers although the principles can also be adapted for graphic equalizers.

Fundamentals

A parametric equalizer is built from a number of filters connected in series (the designs to follow will not work properly if connected in parallel). Each of these filters are called a parametric equalizer filter and are formed as a linear combination of a notch and peak filter, i.e.,

$$H_{\text{eq}}(z) = G_0 H_{\text{notch}}(z) + G H_{\text{peak}}(z), \tag{1}$$

where $H_{\text{notch}}(z)$ and $GH_{\text{peak}}(z)$ are the transfer functions of the notch and peak filters, respectively, and G_0 and G are their gains. More specifically, G is the amplification at the center frequency of the filter relative to G_0. The latter is often chosen as a constant in equalizers and can simply be set to 1, i..e, $G_0 = 1$. Boost is achieved by selecting $G > G_0$ and cut by $G < G_0$. The peak and notch filters share a number of parameters (hence parametric), namely bandwidth $\Delta\omega$, center frequency ω_0, and G, and these can then be adjusted by the user. These digital quantities are related to their analog counterparts as

$$\omega_0 = \frac{2\pi f_0}{f_\mathrm{s}} \quad \text{and} \quad \Delta\omega = \frac{2\pi \Delta f}{f_\mathrm{s}}, \tag{2}$$

where f_s is the sampling frequency and f_0 and Δf are the center frequency and bandwidth (in Hz). In a graphic equalizer, these quantities are fixed for each band and the user can only change G. The center frequency and the bandwidth are related to the edges of the band, termed cutoff frequencies ω_1 and ω_2 (with $\omega_2 > \omega_1$), as

$$\omega_0 = \sqrt{\omega_1 \omega_2} \quad \text{and} \quad \Delta\omega = \omega_2 - \omega_1, \tag{3}$$

i.e., the center frequency is the geometric mean of ω_1 and ω_2 and the bandwidth is simply the difference between the two. Note that the analog counterparts of these quantities, i.e., f_1 and f_2 are related to ω_1 and ω_2 similarly to (2). Note that the Q-factor, which is a quantity that often appears in connection with second-order filters, is given by $Q = \frac{\omega_0}{\Delta\omega}$.

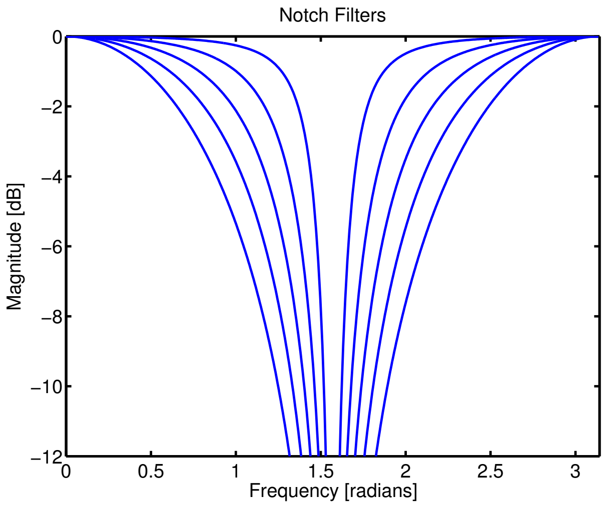

Figure 1: Magnitude responses of notch filters.

Notch Filter

We will now describe the notch filter in (1) in more detail. It serves to attenuate, i.e., cut the signal at a certain frequency, namely the center frequency. A notch filter having a center frequency ω_0 has a transfer function that looks as follows:

$$H_\mathrm{notch}(z) = b \frac{1 - 2\cos\omega_0 z^{-1} + z^{-2}}{1 - 2b\cos\omega_0 z^{-1} + (2b-1)z^{-2}}, \tag{4}$$

which can be seen to be a second-order IIR filter. The quantity b, which is the gain, is given by

$$b = \frac{1}{1+\beta} \quad \text{where} \quad \beta = \frac{\sqrt{1-G_\mathrm{B}^2}}{G_\mathrm{B}} \tan\frac{\Delta\omega}{2}, \tag{5}$$

where G_B is the gain at the cutoff frequencies ω_1 and ω_2. To yield 3 dB cutoff frequencies we could simply choose $G_\mathrm{B}^2 = 0.5$. The above transfer function has been obtained using bilinear z-transformation from an analog equivalent and the frequencies have been pre-warped. The notch filter results in the following difference equation:

$$y_n = bx_n - 2b\cos\omega_0 x_{n-1} + bx_{n-2} + 2b\cos\omega_0 y_{n-1} - (2b-1)y_{n-2}. \tag{6}$$

To implement this, we must simply choose ω_0 and compute b, which is done by selecting a bandwidth $\Delta\omega$. In Figure 1 some examples of magnitude responses of notch filters are shown. These are depicted for a center frequency of $\omega_0 = \pi/2$ and for varying bandwidths with $G_\text{B}^2 = 0.5$.

Peak Filter

Next follows a description of the peak filter in (1). Such a peak filter boosts the signal at a certain frequency, which is the center frequency ω_0 of the filter. It has the following transfer function

$$H_\text{peak}(z) = (1-b)\frac{1-z^{-2}}{1-2b\cos\omega_0 z^{-1} + (2b-1)z^{-2}}, \tag{7}$$

where, similarly to the notch filter, the gain b is given by

$$b = \frac{1}{1+\beta} \quad \text{where} \quad \beta = \frac{G_\text{B}}{\sqrt{1-G_\text{B}^2}}\tan\frac{\Delta\omega}{2}. \tag{8}$$

As can be seen, the peak filter is also a second-order IIR filter. Note that the term z^{-1} does not appear in the numerator of this filter. The parameter $\Delta\omega$ is, just as for the notch filter, the bandwidth of the peak filter at the center frequency ω_0, and G_B is the gain at the cutoff frequencies, which again can be chosen as $G_\text{B}^2 = 0.5$ to yield 3 dB cutoff frequencies. The peak filter is also obtained from an analog filter via the so-called bilinear z-transform and pre-warped frequencies. It has the following difference equation:

$$y_n = (1-b)x_n - (1-b)x_{n-2} + 2b\cos\omega_0 y_{n-1} - (2b-1)y_{n-2}, \tag{9}$$

which can readily be implemented once $\Delta\omega$ and ω_0 have been chosen and b computed from the above expression. In Figure 2, some examples of magnitude responses of peak filters are shown. Shown are the responses for a center frequency of $\omega_0 = \pi/2$ and for varying bandwidths with $G_\text{B}^2 = 0.5$.

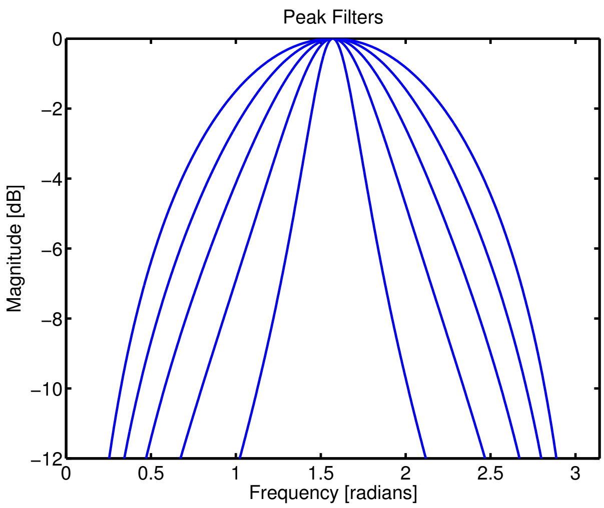

Figure 2: Magnitude responses of peak filters.

Parametric Equalizer Filter

We will now combine the peak and the notch filters that we have just introduced as described in (1). This is done by inserting (4) and (7) into (1). But first, we note that we can combine the two filters as follows: Let

$$H_{\text{notch}}(z) = \frac{A(z)}{B(z)} \tag{10}$$

and

$$H_{\text{peak}}(z) = \frac{C(z)}{D(z)}, \tag{11}$$

then we can combine the two filters into a single filter as

$$\begin{aligned} H_{\text{eq}}(z) &= G_0 H_{\text{notch}}(z) + G H_{\text{peak}}(z) && (12)\\ &= G_0 \frac{A(z)}{B(z)} + G \frac{C(z)}{D(z)} && (13)\\ &= \frac{G_0 A(z) D(z) + G C(z) B(z)}{B(z) D(z)}. && (14) \end{aligned}$$

When combining the two filters, it is interesting to note that

$$H_{\text{notch}}(z) + H_{\text{peak}}(z) = 1, \tag{15}$$

meaning that the two filters with $G_0 = 1$ and $G = 1$ will do nothing to the input signal. The filters are thus said to be complementary. If we choose $G_0 = 0$ and $G = 1$ we obtain a peak filter and, conversely, by setting $G_0 = 1$ and $G = 0$ we obtain a notch filter from our parametric equalizer filter. Hence, the filter is very flexible and contains, as special cases, the peak and notch filters as well as a neutral setting that does nothing at all.

Before proceeding in combining the two filters, we must consider how to choose the gain parameter G_{B}. There are several ways in which this can be done. It is however useful (and common) to set if relative to G and G_0 as either the arithmetic mean or geometric mean of the two, i.e.,

$$G_{\text{B}}^2 = \frac{G^2 + G_0^2}{2} \tag{16}$$

or $G_{\text{B}}^2 = G G_0$. Using the aforementioned principle in combining the two filters, we obtain

$$H_{\text{eq}}(z) = \frac{\frac{G_0 + G\beta}{1+\beta} - 2\frac{G_0 \cos\omega_0}{1+\beta} z^{-1} + \frac{G_0 - G\beta}{1+\beta} z^{-2}}{1 - 2\frac{\cos\omega_0}{1+\beta} z^{-1} + \frac{1-\beta}{1+\beta} z^{-2}}, \tag{17}$$

where β is now defined as

$$\beta = \sqrt{\frac{G_{\text{B}}^2 - G_0^2}{G^2 - G_{\text{B}}^2}} \tan\left(\frac{\Delta\omega}{2}\right). \tag{18}$$

If $G_0^2 < G_{\text{B}}^2 < G^2$ we have a boost at frequency ω_0, and if we have $G^2 < G_{\text{B}}^2 < G_0^2$ we have a cut.

If we use the arithmetic mean in (16) for our definition of G_{B}, it can easily be seen that

$$\sqrt{\frac{G_{\text{B}}^2 - G_0^2}{G^2 - G_{\text{B}}^2}} = 1, \tag{19}$$

in which case we have that $\beta = \tan\left(\frac{\Delta\omega}{2}\right)$. Once we have chosen ω_0 and $\Delta\omega$, it is relatively easy to compute the filters coefficients. The resulting difference equation is given by

$$y_n = \frac{G_0 + G\beta}{1+\beta}x_n - 2\frac{G_0\cos\omega_0}{1+\beta}x_{n-1} + \frac{G_0 - G\beta}{1+\beta}x_{n-2} + 2\frac{\cos\omega_0}{1+\beta}y_{n-1} - \frac{1-\beta}{1+\beta}y_{n-2}. \quad (20)$$

In Figure 3 some examples of magnitude responses of parametric equalizer filters are shown for varying gains, G. The filters shown all have a center frequency of $\omega_0 = \pi/2$, a bandwidth of $\Delta\omega = \pi/4$ with $G_{\mathrm{B}} = \sqrt{GG_0}$ and $G_0 = 1$.

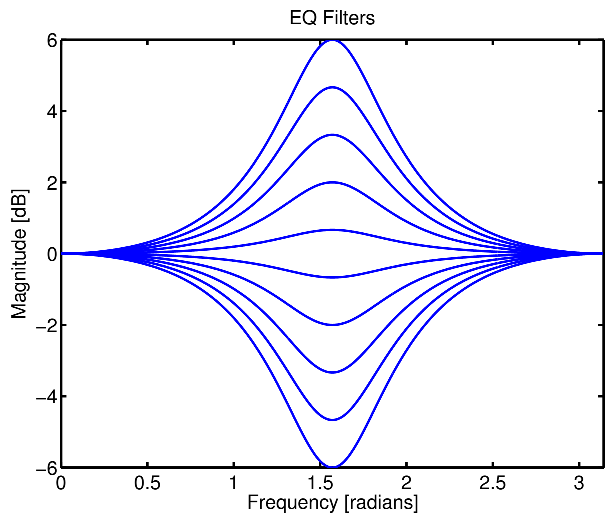

Figure 3: Magnitude responses of parametric equalizer filters.

Shelving Filters

To deal with very low and very high frequencies in the audio signal, it is useful to introduce so-called shelving filters. These are filters that have a flat response and adjustable gain at either low or high frequencies. The filter that is flat for low frequencies is often called a low-pass shelving filter and the one for high frequencies is called high-pass shelving filter. These can be obtained from the general parametric equalizer filter in (17) by replacing the center frequency, ω_0 with either 0 or π.
For the low-pass (LP) shelving filter, we insert $\omega_0 = 0$ into (17). Since $\cos\omega_0 = 1$, we obtain a simpler filter, i.e.,

$$H_{\mathrm{lp}}(z) = \frac{\frac{G_0+G\beta}{1+\beta} - \frac{G_0-G\beta}{1+\beta}z^{-1}}{1 - \frac{1-\beta}{1+\beta}z^{-1}}, \quad (21)$$

which is a first-order filter. Instead of the bandwidth $\Delta\omega$, we now have a single cutoff frequency, which we term ω_{c}, and a corresponding gain G_{C}, which replaces G_{B}. The quantity β in the equations above are then determined from this as

$$\beta = \sqrt{\frac{G_{\mathrm{C}}^2 - G_0^2}{G^2 - G_{\mathrm{C}}^2}}\tan\left(\frac{\omega_{\mathrm{c}}}{2}\right). \quad (22)$$

However, as before, with the definition in (16) for G_{C}, this reduces to something somewhat simpler that only depends on the bandwidth. It should be stressed that the general

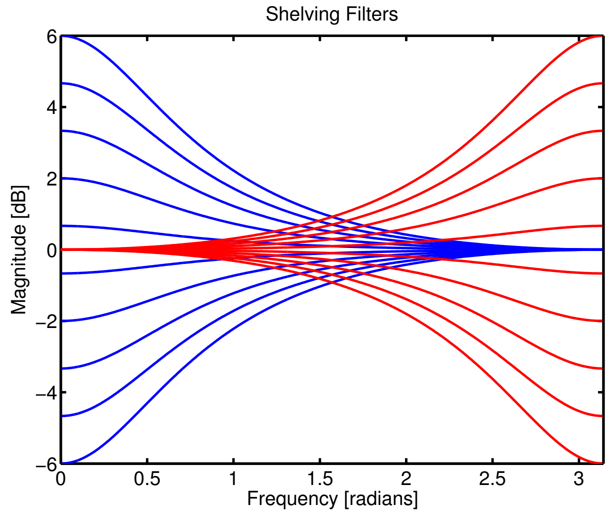

Figure 4: Magnitude responses of low-pass and high-pass shelving filters.

parametric equalizer filter in (17) can be used directly with the substitutions described here. Hence, one does not even have to use a separate implementation of the shelving filters. For the low-pass shelving filter, the resulting difference equation is thus:

$$y_n = \frac{G_0 + G\beta}{1+\beta} x_n - \frac{G_0 - G\beta}{1+\beta} x_{n-1} + \frac{1-\beta}{1+\beta} y_{n-1}. \tag{23}$$

For the high-pass (HP) shelving filter, we insert $\omega_0 = \pi$ into (17). Noting that $\cos\omega_0 = -1$, this yields

$$H_{\text{hp}}(z) = \frac{\frac{G_0+G\beta}{1+\beta} + \frac{G_0-G\beta}{1+\beta} z^{-1}}{1 + \frac{1-\beta}{1+\beta} z^{-1}}, \tag{24}$$

The definition of bandwidth is slightly different int his case, however. Here, it is given by $\Delta\omega = \pi - \omega_{\text{c}}$. Noting that $\tan\left(\frac{\pi-\omega_{\text{c}}}{2}\right) = \cot\left(\frac{\omega_{\text{c}}}{2}\right)$, β turns out, for the high-pass shelving filter, to be

$$\beta = \sqrt{\frac{G_{\text{C}}^2 - G_0^2}{G^2 - G_{\text{C}}^2}} \cot\left(\frac{\omega_{\text{c}}}{2}\right). \tag{25}$$

Finally, the difference equation for the high-pass shelving filter is thus:

$$y_n = \frac{G_0 + G\beta}{1+\beta} x_n + \frac{G_0 - G\beta}{1+\beta} x_{n-1} - \frac{1-\beta}{1+\beta} y_{n-1}. \tag{26}$$

Finally, some examples of magnitude responses of shelving filters are shown in Figure 4 for varying gains, G. Shown are both low-pass and high-pass filters for $\omega_c = \pi/4$ for the low-pass shelving filter and $\omega_c = 3\pi/4$ for the high-pass with $G_{\text{C}} = \sqrt{GG_0}$ and $G_0 = 1$ in both cases.

Summary and Practicalities

All the filters we have considered in this lecture note are special cases of a general 2nd order IIR filter of the form

$$y_n = b_0 x_n + b_1 x_{n-1} + b_2 x_{n-2} + a_1 y_{n-1} + a_2 y_{n-2}. \tag{27}$$

The only difference between the various filters are the choice of the coefficients. A block diagram of the filter structure that can be used for implementing the equalizer filters is

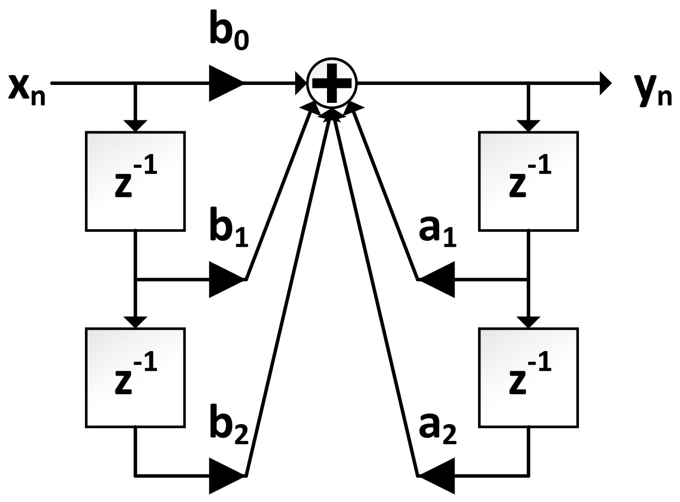

Figure 5: Block diagram of 2nd order filter structure that can be used for implemeting the audio equalizer filters.

Filter Type	b_0	b_1	b_2	a_1	a_2	β
Equalizer	$\frac{G_0+G\beta}{1+\beta}$	$-2\frac{G_0\cos\omega_0}{1+\beta}$	$\frac{G_0-G\beta}{1+\beta}$	$2\frac{\cos\omega_0}{1+\beta}$	$-\frac{1-\beta}{1+\beta}$	$\sqrt{\frac{G_{\mathrm{B}}^2-G_0^2}{G^2-G_{\mathrm{B}}^2}}\tan\left(\frac{\Delta\omega}{2}\right)$
LP shelving	$\frac{G_0+G\beta}{1+\beta}$	$-\frac{G_0-G\beta}{1+\beta}$		$\frac{1-\beta}{1+\beta}$		$\sqrt{\frac{G_{\mathrm{C}}^2-G_0^2}{G^2-G_{\mathrm{C}}^2}}\tan\left(\frac{\omega_{\mathrm{c}}}{2}\right)$
HP shelving	$\frac{G_0+G\beta}{1+\beta}$	$\frac{G_0-G\beta}{1+\beta}$		$-\frac{1-\beta}{1+\beta}$		$\sqrt{\frac{G_{\mathrm{C}}^2-G_0^2}{G^2-G_{\mathrm{C}}^2}}\cot\left(\frac{\omega_{\mathrm{c}}}{2}\right)$

Table 1: General expressions for the filter coefficients for the parametric equalizer filter and the low-pass and high-pass shelving filters for implementation in (27).

shown in Figure 5. In Table 1 the filter coefficients for the various filters are listed along with the appropriate definition of β. If we choose the definition in (16) for G_{B}, i.e., $G_{\mathrm{B}}^2 = \frac{G^2+G_0^2}{2}$, then the various definitions of β become simpler. Moreover, with a choice of unit level, i.e., $G_0 = 1$, the expression are simplified further, and the resulting filter coefficients are listed in Table 2. For quick reference, the user parameters of the various filters are listed in Table 3. It should be noted that instead of the bandwidth, $\Delta\omega$, it is possible to use the so-called Q-factor, which is given by $Q = \frac{\omega_0}{\Delta\omega}$, something that is done in some implementations of parametric equalizers. On a related matter, we have usually specified the parametric equalizer filter in terms of center frequency, ω_0, and bandwidth, $\Delta\omega$. These are related to the cutoff frequencies, ω_1 and ω_2 (where we assume that these have been ordered so that $\omega_1 < \omega_2$), as described in (3). It is possible to determined these from ω_0 and $\Delta\omega$ as follows. Since we have that $\Delta\omega = \omega_2 - \omega_1$, we also have that

$$\omega_0^2 = \omega_1\omega_2 \tag{28}$$

$$= \omega_1\left(\Delta\omega + \omega_1\right) \tag{29}$$

$$= \omega_1^2 + \Delta\omega\omega_1, \tag{30}$$

which means that we can find ω_1 as the values for which

$$\omega_1^2 + \Delta\omega\omega_1 - \omega_0^2 = 0, \tag{31}$$

Filter Type	b_0	b_1	b_2	a_1	a_2	β
Equalizer	$\frac{1+G\beta}{1+\beta}$	$-2\frac{\cos\omega_0}{1+\beta}$	$\frac{1-G\beta}{1+\beta}$	$2\frac{\cos\omega_0}{1+\beta}$	$-\frac{1-\beta}{1+\beta}$	$\tan\left(\frac{\Delta\omega}{2}\right)$
LP shelving	$\frac{1+G\beta}{1+\beta}$	$-\frac{1-G\beta}{1+\beta}$		$\frac{1-\beta}{1+\beta}$		$\tan\left(\frac{\omega_c}{2}\right)$
HP shelving	$\frac{1+G\beta}{1+\beta}$	$\frac{1-G\beta}{1+\beta}$		$-\frac{1-\beta}{1+\beta}$		$\cot\left(\frac{\omega_c}{2}\right)$

Table 2: Simplified expressions for the filter coefficients for the parametric equalizer filter and the low-pass and high-pass shelving filters for implementation in (27) for the case of $G_0 = 1$ and $G_B^2 = \frac{G^2+G_0^2}{2}$.

Filter Type	Parameters	Comment
Notch	ω_0, $\Delta\omega$, G_B	For 3 dB cutoff frequencies set $G_B^2 = 1/2$
Peak	ω_0, $\Delta\omega$, G_B	For 3 dB cutoff frequencies set $G_B^2 = 1/2$
Equalizer	ω_0, $\Delta\omega$, G	Assuming G_B defined from G_0, G
LP shelving	ω_c, G	Obtained by $\omega_0 = 0$ and $\Delta\omega = \omega_c$
HP shelving	ω_c, G	Obtained by $\omega_0 = \pi$ and $\Delta\omega = \pi - \omega_c$

Table 3: List of user parameters for each filter type with G_0 chosen a priori. Note that the bandwidth parameter can be replaced by the Q-factor with $Q = \omega_0/\Delta\omega$.

i.e., ω_1 are the roots of the polynomial. Having found ω_1, we can easily find ω_2 as $\Delta\omega + \omega_1 = \omega_2$. This is useful in case we wish to be able to map bandwidth and center frequencies to cutoff frequencies and vice versa, something that might be useful when matching the cutoff frequencies of adjacent filters.

Literature

[1] S. J. Orfanidis, *Introduction to Signal Processing*. Prentice-Hall International, Inc., 1996.